

# A Loomis-Sikorski theorem for a commutative generalized Hermitian algebra

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## Abstract

A generalized Hermitian (GH-) algebra is a generalization of the partially ordered Jordan algebra of all Hermitian operators on a Hilbert space. We introduce the notion of a gh-tribe, which is a commutative GH-algebra of functions on a nonempty set  $X$  with pointwise partial order and operations, and we prove that every commutative GH-algebra is the image of a gh-tribe under a surjective GH-morphism.

## 1 Introduction

Generalized Hermitian (GH-) algebras, which were introduced in [16] and further studied in [17, 19], incorporate several important algebraic and order theoretic structures including effect algebras [13], MV-algebras [8], orthomodular lattices [30], Boolean algebras [39], and Jordan algebras [33]. Apart from their intrinsic interest, all of the latter structures host mathematical models for quantum-mechanical notions such as observables, states, and experimentally testable propositions [11, 40] and thus are pertinent in regard to the quantum-mechanical theory of measurement [4].

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As indicated by the title, our purpose in this paper is to formulate and prove a version of the Loomis-Sikorski theorem for commutative GH-algebras (Theorem 6.6 below). This theorem is a generalization of the classical Loomis-Sikorski theorem for Boolean  $\sigma$ -algebras. The classical theorem plays an important role in the definition of the functional calculus for compatible observables on orthomodular lattices [40, Theorem 3.9], and in subsequent investigations, we shall use our more general theorem to assist in the formulation of a theory of observables and functional calculus for GH-algebras.

It turns out that GH-algebras are special cases of the more general synaptic algebras introduced in [14], and further studied in [18, 20, 21, 22, 24, 25, 26, 27, 37]. Thus, in this paper, it will be convenient for us to treat GH-algebras as special kinds of synaptic algebras (Section 3 below).

## 2 Preliminaries

In this section we review some notions and some facts that will be needed as we proceed. We abbreviate ‘if and only if’ as ‘iff,’ the notation  $:=$  means ‘equals by definition,’  $\mathbb{R}$  is the ordered field of real numbers,  $\mathbb{R}^+ := \{\alpha \in \mathbb{R} : 0 \leq \alpha\}$ , and  $\mathbb{N} := \{1, 2, 3, \dots\}$  is the well-ordered set of natural numbers.

**2.1 Definition.** Let  $\mathcal{P}$  be a partially ordered set (poset). Then:

- (1) Let  $p, q \in \mathcal{P}$ . Then an existing supremum, i.e. least upper bound, (an existing infimum, i.e., greatest lower bound) of  $p$  and  $q$  in  $\mathcal{P}$  is written as  $p \vee q$  ( $p \wedge q$ ). If it is necessary to make clear that the supremum (infimum) is calculated in  $\mathcal{P}$ , we write  $p \vee_{\mathcal{P}} q$  ( $p \wedge_{\mathcal{P}} q$ ).  $\mathcal{P}$  is a *lattice* iff  $p \vee q$  and  $p \wedge q$  exist for all  $p, q \in \mathcal{P}$ . If  $\mathcal{P}$  is a lattice, then a nonempty subset  $\mathcal{Q} \subseteq \mathcal{P}$  is a *sublattice* of  $\mathcal{P}$  iff, for all  $p, q \in \mathcal{Q}$ ,  $p \vee q, p \wedge q \in \mathcal{Q}$ , in which case  $\mathcal{Q}$  is a lattice in its own right with  $p \vee_{\mathcal{Q}} q = p \vee q$  and  $p \wedge_{\mathcal{Q}} q = p \wedge q$ .
- (2) A lattice  $\mathcal{P}$  is *distributive* iff, for all  $p, q, r \in \mathcal{P}$ ,  $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ , or equivalently,  $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ .
- (3) The poset  $\mathcal{P}$  is *bounded* iff there are elements, usually denoted by 0 and 1, such that  $0 \leq p \leq 1$  for all  $p \in \mathcal{P}$ . If  $\mathcal{P}$  is a bounded lattice then elements  $p, q \in \mathcal{P}$  are *complements* of each other iff  $p \wedge q = 0$  and  $p \vee q = 1$ . A *Boolean algebra* is a bounded distributive lattice in

which every element has a complement. In a Boolean algebra  $\mathcal{P}$ , the complement of an element  $p$  is unique, and is often denoted by  $p'$ .

- (3)  $\mathcal{P}$  is  $\sigma$ -complete iff every sequence in  $\mathcal{P}$  has both a supremum and an infimum in  $\mathcal{P}$ .
- (4)  $\mathcal{P}$  is *Dedekind  $\sigma$ -complete* iff every sequence in  $\mathcal{P}$  that is bounded above (below) has a supremum (an infimum) in  $\mathcal{P}$ .
- (5) If  $p_1 \leq p_2 \leq p_3 \leq \cdots$  is an ascending sequence in  $\mathcal{P}$  with supremum  $p = \bigvee_{n=1}^{\infty} p_n$  in  $\mathcal{P}$ , we write  $p_n \nearrow p$ . Similar notation  $p_n \searrow p$  applies to a descending sequence  $p_1 \geq p_2 \geq p_3 \geq \cdots$  with infimum  $p = \bigwedge_{n=1}^{\infty} p_n$  in  $\mathcal{P}$ .
- (6)  $\mathcal{P}$  is *monotone  $\sigma$ -complete* iff, for every bounded ascending (descending) sequence  $(p_n)_{n=1}^{\infty}$  in  $\mathcal{P}$ , there exists  $p \in G$  with  $p_n \nearrow p$  ( $p_n \searrow p$ ).

**2.2 Remarks.** An *involution* on the poset  $\mathcal{P}$  is a mapping  $': \mathcal{P} \rightarrow \mathcal{P}$  such that, for all  $p, q \in \mathcal{P}$ ,  $p \leq q \Rightarrow q' \leq p'$  and  $(p')' = p$ . For instance, the complementation mapping  $p \mapsto p'$  on a Boolean algebra  $\mathcal{P}$  is an involution. An involution on a poset  $\mathcal{P}$  provides a “duality” between existing suprema and infima of subsets  $\mathcal{Q}$  of  $\mathcal{P}$  as follows: If the supremum  $\bigvee \mathcal{Q}$  (the infimum  $\bigwedge \mathcal{Q}$ ) exists, then the infimum  $\bigwedge \{q' : q \in \mathcal{Q}\}$  (the supremum  $\bigvee \{q' : q \in \mathcal{Q}\}$ ) exists and equals  $(\bigvee \mathcal{Q})'$  (equals  $(\bigwedge \mathcal{Q})'$ ). Thus, if  $\mathcal{P}$  admits an involution, then condition (3) in Definition 2.1 is equivalent to the same condition for sequences bounded above (below) only, and similar remarks hold for conditions (4) and (6).

Recall that an *order unit normed space*  $(V, u)$  [1, pp. 67–69] is a partially ordered linear space  $V$  over  $\mathbb{R}$  with positive cone  $V^+ = \{v \in V : 0 \leq v\}$  such that: (1)  $V$  is *archimedean*, i.e., if  $v \in V$  and  $\{nv : n \in \mathbb{N}\}$  is bounded above in  $V$ , then  $-v \in V^+$ . (2)  $u \in V^+$  is an *order unit* (sometimes called a *strong order unit* [38]), i.e., for each  $v \in V$  there exists  $n \in \mathbb{N}$  such that  $v \leq nu$ . Then the *order-unit norm* on  $(V, u)$  is defined by  $\|v\| = \inf\{\lambda \in \mathbb{R}^+ : -\lambda u \leq v \leq \lambda u\}$  for all  $v \in V$ . If  $(V, u)$  is an order unit normed space, then the mapping  $v \mapsto -v$  is an involution on  $V$ , so Remarks 2.2 apply. If  $V$  is an archimedean partially ordered real linear space and  $u \in V^+$  is an order unit, then—if it is understood  $u$  is the order unit in question—we may simply say that  $V$  (rather than  $(V, u)$ ) is an order unit normed space.

See [1, Proposition 2.I.2] and [28, Proposition 7.12.(c)], for a proof of the following.

**2.3 Lemma.** *If  $(V, u)$  is an order unit normed space, then for all  $v, w \in V$ , (i)  $-\|v\|u \leq v \leq \|v\|u$  and (ii)  $-w \leq v \leq w \Rightarrow \|v\| \leq \|w\|$ .*

The next theorem [23] follows, *mutatis mutandis*, from D. Handelman's proof of [29, Proposition 3.9].

**2.4 Theorem.** *If  $(V, u)$  is an order unit normed space such that  $V$  is monotone  $\sigma$ -complete, then  $V$  is a Banach space under the order-unit norm.*

Let  $(V, u)$  be an order unit normed space. Then elements of the “unit interval”  $V[0, u] := \{e \in V : 0 \leq e \leq u\}$  are called *effects*, and  $V[0, u]$  is organized into a so-called *effect algebra*  $(V[0, u]; 0, u, ^\perp, \oplus)$  [13] as follows: For  $e, f \in V[0, u]$ ,  $e \oplus f$  is defined iff  $e + f \leq u$ , and then  $e \oplus f := e + f$ ; moreover,  $e^\perp := u - e$ . The effect algebra  $V[0, u]$  is partially ordered by the relation  $e \leq f$  iff there is  $g \in V[0, u]$  such that  $e \oplus g = f$ , and this ordering coincides with that inherited from  $V$ . The mapping  $e \mapsto e^\perp$  is an involution on  $V[0, u]$ , so Remarks 2.2 apply. With the convexity structure induced by the linearity of  $V$ ,  $V[0, u]$  is a convex effect algebra [5, 6]. Two effects  $e, f \in V[0, u]$  are said to be (*Mackey*) *compatible* iff there are elements  $e_1, f_1, d \in V[0, u]$  such that  $e_1 + f_1 + d \leq u$ ,  $e = e_1 + d$ , and  $f = f_1 + d$ .

An effect algebra that forms a lattice is said to be *lattice ordered*, or simply a *lattice effect algebra*. A lattice effect algebra in which every pair of elements are compatible is called an *MV-effect algebra*. As a lattice, an MV-effect algebra is distributive. It is known that MV-effect algebras are mathematically equivalent to MV-algebras, which were introduced by Chang [8] as algebraic bases for many-valued logics. (See, e.g., [11] for the relations between lattice effect algebras and MV-algebras). Notice that a convex effect algebra is an MV-effect algebra iff it is a lattice [5]. By a well-known result of D. Mundici [35], every MV-algebra is isomorphic to the unit interval  $G[0, u]$  in a lattice ordered abelian group (abelian  $\ell$ -group)  $G$  with order unit  $u$ . An MV-algebra that is a  $\sigma$ -complete lattice is called a  $\sigma$ MV-algebra [10], and it turns out that an MV-algebra is a  $\sigma$ MV-algebra iff the corresponding abelian  $\ell$ -group is Dedekind  $\sigma$ -complete.

A partially ordered linear space  $V$  over  $\mathbb{R}$  that is a lattice under the partial order is called a *vector lattice* or a *Riesz space*. Every vector lattice, and more generally, every abelian  $\ell$ -group, is distributive [3, Theorem 4]. If  $V$  is a vector lattice and  $v \in V$ , then the *absolute value* and the *positive part* of  $v$  are denoted and defined by  $|v| := v \vee (-v)$  and  $v^+ := v \vee 0$ , respectively. A vector lattice  $V$  satisfies *Dedekind's law*: For  $v, w \in V$ ,  $v \vee w + v \wedge w = v + w$ .

**2.5 Lemma.** *If  $V$  is a vector lattice, then  $V$  is monotone  $\sigma$ -complete iff  $V$  is Dedekind  $\sigma$ -complete.*

*Proof.* Suppose that  $V$  is monotone  $\sigma$ -complete, let  $(v_n)_{n \in \mathbb{N}}$  be a sequence in  $V$  that is bounded above, and define  $(w_n)_{n \in \mathbb{N}}$  by  $w_n := v_1 \vee v_2 \vee \cdots \vee v_n$  for all  $n \in \mathbb{N}$ . Then,  $(w_n)_{n \in \mathbb{N}}$  is an ascending sequence in  $V$  with the same set of upper bounds as  $(v_n)_{n \in \mathbb{N}}$ , whence  $w_n \nearrow w \in V$ , and  $w = \bigvee_{n \in \mathbb{N}} v_n$ . Thus  $V$  is Dedekind  $\sigma$ -complete. The converse is obvious.  $\square$

Of course, by an *order unit normed vector lattice*, we mean an order unit normed space that is also a vector lattice.

**2.6 Theorem.** *If  $V$  is a monotone  $\sigma$ -complete vector lattice with order unit  $u$ , then  $(V, u)$  is an order unit normed vector lattice and a Banach space under the order-unit norm.*

*Proof.* By [9, 3.2.5 Prop. 2], every monotone  $\sigma$ -complete vector lattice is archimedean. By Theorem 2.4, every monotone  $\sigma$  complete order unit normed space is Banach.  $\square$

**2.7 Lemma.** *Let  $(V, u)$  be an order unit normed vector lattice and let  $v, w \in V$ . Then: (i)  $0 < \epsilon \in \mathbb{R} \Rightarrow |w| \leq \epsilon u \Leftrightarrow \|w\| \leq \epsilon$ . (ii) If  $(v_n)_{n \in \mathbb{N}}$  is a sequence in  $V$ , then  $\lim_{n \rightarrow \infty} v_n = v$  iff, for every  $0 < \epsilon \in \mathbb{R}$ , there exists  $N \in \mathbb{N}$  such that, for all  $n \in \mathbb{N}$ ,  $N \leq n \Rightarrow |v_n - v| \leq \epsilon u$ .*

*Proof.* Let  $0 < \epsilon \in \mathbb{R}$  and  $v, w \in V$ . (i) By Lemma 2.3 (i),  $w, -w \leq \|w\|u$ , whence  $\|w\| \leq \epsilon \Rightarrow w, -w \leq \epsilon u \Rightarrow -\epsilon u \leq w \leq \epsilon u$ . Conversely,  $-\epsilon u \leq w \leq \epsilon u \Rightarrow \|w\| \leq \epsilon$ , whence  $-\epsilon u \leq w \leq \epsilon u \Leftrightarrow \|w\| \leq \epsilon$ . Thus, since  $|w| = w \vee (-w)$ , we have  $|w| \leq \epsilon u \Leftrightarrow w, -w \leq \epsilon u \Leftrightarrow -\epsilon u \leq w \leq \epsilon u \Leftrightarrow \|w\| \leq \epsilon$ . (ii) Putting  $w = v_n - v$  in (i), we have  $|v_n - v| \leq \epsilon u \Leftrightarrow \|v_n - v\| \leq \epsilon$ , from which (ii) follows.  $\square$

Evidently, if  $(V, u)$  is a Dedekind  $\sigma$ -complete order unit normed vector lattice, then the unit interval  $V[0, u]$  is a convex  $\sigma$ MV-algebra. And conversely, by [5, 35], every convex  $\sigma$ MV-algebra is isomorphic to the unit interval in a Dedekind  $\sigma$ -complete order unit normed vector lattice.

Let  $(V, u)$  be an order unit normed vector lattice. An element  $e \in V$  is called a *characteristic element* [28, Definition p. 127] (or a *unitary element* [9, Definition 4.1.1.2.]) iff  $e \wedge (u - e) = 0$ . The set  $B$  of all characteristic elements in  $V$  is a sub-effect algebra of  $V[0, u]$ ; moreover,  $B$  is a sublattice

of  $V$ , and as such  $B$  is a Boolean algebra with  $u - e$  as the complement of  $e \in B$  [9, Remark p. 120]. Two elements  $e, f \in B$  are said to be *orthogonal* iff  $e \wedge f = 0$ , or equivalently, iff  $e \leq u - f$ . By Dedekind's law, if  $e$  and  $f$  are orthogonal elements of  $V$ , then  $e \vee f = e + f$ . An element in  $V$  is said to be *simple* iff it is a finite real linear combination of characteristic elements. By an adaptation of the proof of [15, Theorem 5.1], it can be shown that every simple element in  $V$  can be written as a finite real linear combination of pairwise orthogonal characteristic elements.

Let  $X$  be a compact Hausdorff space. We define  $\mathcal{F}(X)$  to be the field of all compact open (clopen) subsets of  $X$ , noting that  $\mathcal{F}(X)$  is a Boolean algebra under set containment  $\subseteq$ . Also, as usual,  $C(X, \mathbb{R})$  denotes the partially ordered commutative associative real unital Banach algebra of all continuous functions  $f: X \rightarrow \mathbb{R}$ , with pointwise partial order and pointwise finitary operations. Then, with the constant function  $x \mapsto 1$  (denoted simply by 1) as order unit,  $C(X, \mathbb{R})$  is an order unit normed vector lattice, and the order-unit norm coincides with the supremum (or uniform) norm on  $C(X, \mathbb{R})$ . For  $f \in C(X, \mathbb{R})$ ,  $|f| = f \vee (-f)$  is the pointwise absolute value, and it follows from Lemma 2.7 (ii) that order-unit norm limits of sequences in  $C(X, \mathbb{R})$  are pointwise limits.

We denote by  $P(X, \mathbb{R})$  the Boolean algebra of all characteristic elements in  $C(X, \mathbb{R})$ . It is not difficult to see that  $P(X, \mathbb{R}) = \{p \in C(X, \mathbb{R}) : p = p^2\}$  and also that  $P(X, \mathbb{R})$  consists of all characteristic set functions (indicator functions)  $\chi_K$  of clopen sets  $K \in \mathcal{F}(X)$ , whence the Boolean algebra  $P(X, \mathbb{R})$  is isomorphic to  $\mathcal{F}(X)$  under the mapping  $\chi_K \mapsto K$ .

A *Stone space* is a compact Hausdorff space  $X$  such that the clopen sets in  $\mathcal{F}(X)$  form a basis for the open sets in  $X$ . If  $X$  is a Stone space, and  $x, y \in X$  with  $x \neq y$ , then there are disjoint clopen sets  $K, L \in \mathcal{F}(X)$  with  $x \in K$  and  $y \in L$ ; hence  $\chi_K(x) = 1$  and  $\chi_{X \setminus L}(y) = 0$ . Therefore distinct points in  $X$  are separated by continuous functions in  $P(X, \mathbb{R}) \subseteq C(X, \mathbb{R})$ .

By the *Stone representation theorem*, if  $\mathcal{P}$  is a Boolean algebra, there is a Stone space  $X$ , called *the Stone space of  $\mathcal{P}$*  and uniquely determined up to a homeomorphism, such that  $\mathcal{P}$  is isomorphic to the Boolean algebra  $\mathcal{F}(X)$ , whence  $\mathcal{P}$  is isomorphic to  $P(X, \mathbb{R})$ . As is well-known,  $\mathcal{P}$  is a  $\sigma$ -complete Boolean algebra (a Boolean  $\sigma$ -algebra) iff the Stone space  $X$  of  $\mathcal{P}$  is basically disconnected, i.e., the closure of every open  $F_\sigma$  subset of  $X$  remains open.

By [9, Theorem 2, p. 150] and Theorem 2.5, we have the following.

**2.8 Theorem.** *Let  $(V, u)$  be a monotone  $\sigma$ -complete order unit normed vector*

lattice, let  $B$  be the Boolean  $\sigma$ -algebra of characteristic elements in  $V$ , and let  $X$  be the basically disconnected Stone space of  $B$ . Then: (i)  $V$  is Dedekind  $\sigma$ -complete. (ii) There is an isomorphism of order unit normed vector lattices,  $\Psi: V \rightarrow C(X, \mathbb{R})$ , of  $V$  onto  $C(X, \mathbb{R})$  such that the restriction  $\psi$  of  $\Psi$  to  $B$  is a Boolean isomorphism of  $B$  onto  $P(X, \mathbb{R})$  as per Stone's theorem.

### 3 Synaptic algebras and generalized Hermitian algebras

The axioms SA1–SA8 for a synaptic algebra appear in the following definition [14, Definition 1.1]. To help fix ideas before we proceed, we remark that the system  $A = \mathcal{B}^{sa}(\mathfrak{H})$  of all bounded self-adjoint linear operators on a Hilbert space  $\mathfrak{H}$  with the algebra  $R = \mathcal{B}(\mathfrak{H})$  of all bounded linear operators on  $\mathfrak{H}$  as its enveloping algebra is a prototypic example of a synaptic algebra.

**3.1 Definition.** Let  $R$  be a linear associative algebra over the real or complex numbers with unity element 1 and let  $A$  be a real vector subspace of  $R$ . If  $a, b \in A$ , we understand that the product  $ab$ , which may or may not belong to  $A$ , is calculated in  $R$ , and we say that  $a$  *commutes with*  $b$ , in symbols  $aCb$ , iff  $ab = ba$ . If  $a \in A$  and  $B \subseteq A$ , we define  $C(a) := \{b \in A : aCb\}$ ,  $C(B) := \bigcap_{b \in B} C(b)$ ,  $CC(B) := C(C(B))$ , and  $CC(a) := C(C(a))$ . Of course,  $B$  is said to be *commutative* iff  $aCb$  for all  $a, b \in B$ , i.e., iff  $B \subseteq C(B)$ .

The vector space  $A$  is a *synaptic algebra* with *enveloping algebra*  $R$  iff the following conditions are satisfied:

- SA1.  $(A, 1)$  is an order unit normed space with positive cone  $A^+ := \{a \in A : 0 \leq a\}$  and  $\|\cdot\|$  is the corresponding order-unit norm on  $A$ .
- SA2. If  $a \in A$  then  $a^2 \in A^+$ .
- SA3. If  $a, b \in A^+$ , then  $aba \in A^+$ .
- SA4. If  $a \in A$  and  $b \in A^+$ , then  $aba = 0 \Rightarrow ab = ba = 0$ .
- SA5. If  $a \in A^+$ , there exists  $b \in A^+ \cap CC(a)$  such that  $b^2 = a$ .
- SA6. If  $a \in A$ , there exists  $p \in A$  such that  $p = p^2$  and, for all  $b \in A$ ,  $ab = 0 \Leftrightarrow pb = 0$ .

SA7. If  $1 \leq a \in A$ , there exists  $b \in A$  such that  $ab = ba = 1$ .

SA8. If  $a, b \in A$ ,  $a_1 \leq a_2 \leq a_3 \leq \dots$  is an ascending sequence of pairwise commuting elements of  $C(b)$  and  $\lim_{n \rightarrow \infty} \|a - a_n\| = 0$ , then  $a \in C(b)$ .

For the remainder of this paper we assume that  $A$  is a synaptic algebra. We shall also assume that  $1 \neq 0$  (i.e.,  $A \neq \{0\}$ ), which enables us (as usual) to identify each real number  $\lambda \in \mathbb{R}$  with  $\lambda 1 \in A$ . Thus, for  $a \in A$ ,  $\|a\| = \inf\{0 < \lambda \in \mathbb{R} : -\lambda \leq a \leq \lambda\}$ . Limits in  $A$  are understood to be limits with respect to the order-unit norm  $\|\cdot\|$ . In the remainder of this section we comment briefly on some of the basic consequences of SA1–SA8. See [14] for proofs and more details.

If  $a \in A$ , then by SA2,  $a^2 \in A^+ \subseteq A$ . Consequently,  $A$  is a special Jordan algebra with the Jordan product

$$a \odot b := \frac{1}{2}(ab + ba) = \frac{1}{4}[(a + b)^2 - (a - b)^2] \in A \text{ for all } a, b \in A.$$

Thus, if  $a, b \in A$  and  $aCb$ , then  $ab = ba = a \odot b \in A$ . Also,  $\|a \odot b\| \leq \|a\|\|b\|$ .

By a simple calculation,  $aba = 2a \odot (a \odot b) - a^2 \odot b \in A$ , and the mapping  $b \mapsto aba$ , called the *quadratic mapping* determined by  $a$ , turns out to be linear and order preserving. In particular,  $A$  satisfies the condition  $a \in A$ ,  $b \in A^+ \Rightarrow aba \in A^+$ , which is stronger than SA3. By SA4 with  $b = 1$ , we have  $a^2 = 0 \Rightarrow a = 0$ .

Using SA5 and SA2, one can prove that, if  $a, b \in A^+$  and  $aCb$ , then  $ab \in A^+$  [14, Lemma 1.5]. Also, using SA5, and arguing as in [14, Theorem 2.2], it can be shown that every  $a \in A^+$ , has a *square root*  $a^{1/2} \in A^+$  which is uniquely determined by the condition  $(a^{1/2})^2 = a$ ; moreover,  $a^{1/2} \in CC(a)$ .

Let  $a \in A$ . Then  $a^2 \in A^+$  by SA2, and the *absolute value* of  $a$ , defined by  $|a| := (a^2)^{1/2}$ , has the property that  $|a| \in CC(a)$ . Moreover,  $|a|$  is uniquely determined by the properties  $|a| \in A^+$  and  $|a|^2 = a^2$ . Furthermore, using the absolute value of  $a$ , we define the *positive part* and the *negative part* of  $a$  by  $a^+ := \frac{1}{2}(|a| + a) \in A^+ \cap CC(a)$  and  $a^- := (-a)^+ = \frac{1}{2}(|a| - a) \in A^+ \cap CC(a)$ , respectively. Then  $a = a^+ - a^-$ ,  $|a| = a^+ + a^-$ ,  $a^+a^- = 0 = a^-a^+$ . (See Lemma 4.2 below.)

As in Section 2, we define  $E := A[0, 1] = \{e \in A : 0 \leq e \leq 1\}$  and organize  $E$  into a convex effect algebra. An important subset of  $E$  is the set  $P := \{p \in A : p^2 = p\}$  of idempotents in  $A$ , which are called *projections*. By [14, Theorem 2.6 (iii) and (v)] we have the following.



**3.2 Lemma.** *If  $p \in E$ , then the following conditions are mutually equivalent:*  
(i)  $p \in P$ . (ii)  $p$  is an extreme point of  $E$ . (iii)  $p \wedge_E (1 - p) = 0$ .

With the partial order inherited from  $A$  and the orthocomplementation  $p \mapsto p^\perp := 1 - p$ ,  $P$  is an orthomodular lattice (OML) [30] with smallest element 0 and largest element 1. Two projections  $p, q \in P$  are said to be *orthogonal*, in symbols  $p \perp q$ , iff  $p + q \leq 1$ , or equivalently, iff  $pq = qp = 0$ . Also, two elements  $p, q$  in an OML are (*Mackey*) *compatible* [32] iff  $p = (p \wedge q) \vee (p \wedge q^\perp)$  (then automatically  $q = (p \wedge q) \vee (p^\perp \wedge q)$ ). It turns out that two projections  $p$  and  $q$  are compatible in  $P$  iff they are compatible in  $E$  iff  $pCq$ . It is well-known that an OML is a Boolean algebra iff its elements are pairwise compatible.

**3.3 Lemma.** *Let  $p, q \in P$  with  $pCq$ . Then:* (i)  $pq = qp \in P$ . (ii)  $pq \leq p, q$ . (iii)  $p \leq q \Leftrightarrow p = pq$ . (iv)  $p \wedge q = pq$ .

*Proof.* (i) We have  $p = p^2$ ,  $q = q^2$  and  $pCq$  whence  $(pq)^2 = pqpq = p^2q^2 = pq$ , and therefore  $pq = qp \in P$ . (ii) Since  $0 \leq p, 1 - q$  and  $pC(1 - q)$  it follows that  $0 \leq p(1 - q) = p - pq$ , so  $pq \leq p$ , and similarly  $pq \leq q$ . (iii) Suppose that  $p \leq q$ . Then as  $qC(q - p)$ , and  $0 \leq q - p, 1 - q$ , it follows that  $0 \leq (q - p)(1 - q) = pq - p$ , so  $p \leq pq$ . Conversely, by (ii), if  $p = pq$ , then  $p \leq q$ , proving (iii). (iv) By (ii),  $pq$  is a lower bound in  $P$  for  $p$  and  $q$ . Suppose  $r \in P$  and  $r \leq p, q$ . Then  $r = rp = rq$  by (iii), so  $r(pq) = (rp)q = rq = r$ , whence  $r \leq pq$ , proving (iv).  $\square$

An element in  $A$  is called *simple* iff it is a finite linear combination of pairwise commuting projections. Every simple element in  $A$  can be written as a finite linear combination of pairwise orthogonal projections. It turns out that each element  $a \in A$  is the norm limit and also the supremum of an ascending sequence of pairwise commuting simple elements [14, Corollary 8.6 and Theorem 8.9].

Using SA6, one can prove that if  $a \in A$ , there exists a unique projection, denoted by  $a^\circ \in P$  and called the *carrier* of  $a$ , such that, for all  $b \in A$ ,  $ab = 0 \Leftrightarrow a^\circ b = 0$ . (Some authors refer to  $a^\circ$  as the *support* of  $a$ .) It turns out that  $ab = 0 \Leftrightarrow a^\circ b^\circ = 0 \Leftrightarrow b^\circ a^\circ = 0 \Leftrightarrow ba = 0$ . By [14, Theorem 2.10],  $a^\circ \in CC(a)$ ,  $|a|^\circ = a^\circ$ , and  $(a^n)^\circ = a^\circ$  for all  $n \in \mathbb{N}$ .

An element  $a \in A$  is *invertible* iff there is a (necessarily unique) element  $a^{-1} \in A$  such that  $aa^{-1} = a^{-1}a = 1$ . Using SA7, one can prove that  $a \in A$  is invertible iff there exists  $0 < \epsilon \in \mathbb{R}$  such that  $\epsilon \leq |a|$  [14, Theorem 7.2].

In the presence of the remaining axioms, SA8 is equivalent to the condition that  $C(a)$  is norm closed for all  $a \in A$ .

An element  $s \in A$  such that  $s^2 = 1$  is called a *symmetry* [21]. Symmetries are in one-to-one correspondence with projections as follows: If  $s$  is a symmetry, then  $p = \frac{1}{2}(s + 1)$  is a projection, and if  $p$  is a projection, then  $s = 2p - 1$  is a symmetry. If  $s \in A$  is a symmetry, then  $|s| = \|s\| = 1$ .

A subset  $B$  of  $A$  is a *sub-synaptic algebra* iff it is a linear subspace of  $A$ ,  $1 \in B$ , and  $B$  is closed under the formation of squares, square roots, carriers, and inverses. If  $B$  is a sub-synaptic algebra of  $A$ , then  $B$  is a synaptic algebra in its own right under the restrictions to  $B$  of the partial order and operations on  $A$ . For instance, the *center* of  $A$ , i.e., the subset  $C(A)$ , is a sub-synaptic algebra of  $A$ , and it is a commutative synaptic algebra in its own right.

**3.4 Definition.**  $A$  has the *commutative Vigier* (CV) property iff, for every bounded ascending sequence  $(a_n)_{n=1}^\infty$  of pairwise commuting elements in  $A$ , there exists  $a \in CC(\{a_n : n \in \mathbb{N}\})$  with  $a_n \nearrow a$ .

As a consequence of [16, Lemma 6.6] condition CV implies SA8.

**3.5 Remarks.** In view of the discussion in [14, §6], *in what follows, we can and shall regard a GH-algebra as a synaptic algebra in which axiom SA8 is replaced by the stronger CV condition.* By [16, Lemma 5.4], if  $A$  is a GH-algebra, then  $P$  is a  $\sigma$ -complete OML.

We note that  $\mathcal{B}^{sa}(\mathfrak{H})$  is a GH-algebra. Every synaptic algebra of finite rank (meaning that there exists  $n \in \mathbb{N}$  such that there are  $n$ , but not  $n + 1$  pairwise orthogonal nonzero projections in  $P$ ) is a GH-algebra. According to [17], a synaptic algebra of rank 2 is the same thing as a spin factor of dimension greater than 1. Thus, GH-algebras of finite rank need not be finite dimensional. Additional examples of GH-algebras can be found in [14, 19].

Condition (4) in the following definition is an enhancement of [18, Definition 2.9] but this does not affect the definition of a synaptic isomorphism.

**3.6 Definition.** Let  $A_1$  and  $A_2$  be synaptic algebras. A linear mapping  $\phi: A_1 \rightarrow A_2$  is a *synaptic morphism* iff, for all  $a, b \in A_1$ :

- (1)  $\phi(1) = 1$ .                      (2)  $\phi(a^2) = \phi(a)^2$ .
- (3)  $aCb \Rightarrow \phi(a)C\phi(b)$ .      (4)  $\phi(a^\circ) = \phi(a)^\circ$ .

A synaptic morphism  $\phi$  is a *synaptic isomorphism* iff it is a bijection and  $\phi^{-1}$  is also a synaptic morphism. If  $A_1$  and  $A_2$  are GH-algebras, then a *GH-algebra morphism*  $\phi: A_1 \rightarrow A_2$  is a synaptic morphism such that, for every ascending sequence of pairwise commuting elements  $(a_n)_{n=1}^\infty$  in  $A$ ,

$$(5) \ a_n \nearrow a \in A \Rightarrow \phi(a_n) \nearrow \phi(a).$$

A GH-algebra morphism is a *GH-algebra isomorphism* iff it is a bijection and its inverse is also a GH-algebra morphism.

**3.7 Remark.** Since a synaptic isomorphism is necessarily an order isomorphism, if  $A_1$  is a GH-algebra,  $A_2$  is a synaptic algebra, and there is a synaptic isomorphism  $\phi$  from  $A_1$  onto  $A_2$ , then  $A_2$  is a GH-algebra and  $\phi$  is a GH-isomorphism.

## 4 Commutative synaptic and GH-algebras

By [27, Theorem 5.12], we have the following characterization of commutative synaptic algebras.

**4.1 Theorem.** *The following conditions are mutually equivalent: (i) The synaptic algebra  $A$  is commutative. (ii)  $A$  is lattice ordered, hence an order unit normed vector lattice. (iii)  $E$  is an MV-effect algebra. (iv)  $P$  is a Boolean algebra. Moreover, every Boolean algebra can be realized as the Boolean algebra of projections in a commutative synaptic algebra.*

**4.2 Lemma.** *Let  $A$  be commutative. Then: (i) The vector-lattice notions of absolute value and positive part coincide with the corresponding notions as defined for a synaptic algebra. (ii) For  $a, b \in A$ ,  $a \wedge b = \frac{1}{2}(a + b - |a - b|)$  and  $a \vee b = \frac{1}{2}(a + b + |a - b|)$ . (iii)  $P$  is precisely the set of characteristic elements in  $A$ . (iv) The simple elements  $a \in A$  in the sense of an order unit normed vector lattice are precisely the simple elements in  $A$  as defined for a synaptic algebra.*

*Proof.* Part (i) follows from [27, Remarks 5.5] and (ii) is a consequence of [27, §4 and Corollary 5.13]. To prove (iii), we note that if  $p$  is a characteristic element in  $A$ , then  $p \wedge_A (1 - p) = 0$ , whence  $p \wedge_E (1 - p) = 0$ , so  $p \in P$  by Lemma 3.2. Conversely, suppose that  $p \in P$ . Then by (ii),  $p \wedge_A (1 - p) = \frac{1}{2}(p + 1 - p - |p - (1 - p)|) = \frac{1}{2}(1 - |2p - 1|)$ . But,  $2p - 1$  is a symmetry, whence  $|2p - 1| = 1$ , so  $p \wedge_A (1 - p) = 0$ , proving (iii). Part (iv) follows from (iii).  $\square$

Clearly, the synaptic algebra  $A$  is commutative iff it is equal to its own center, i.e.,  $A = C(A)$ . From the results in Section 3, Theorem 4.1, and Lemma 4.2 we have the following: If  $A$  is a commutative synaptic algebra, then  $A$  is a commutative, associative, partially ordered archimedean real linear algebra with a unity element 1; it is an order unit normed vector lattice with order unit 1; it is a normed linear algebra under the order-unit norm; it can be considered as its own enveloping algebra; the characteristic elements in  $A$  coincide with the projections in the Boolean algebra  $P$ ; and the set of simple elements in  $A$  is a norm dense commutative sub-synaptic algebra of  $A$ .

If  $A$  is commutative, then the CV condition reduces to monotone  $\sigma$ -completeness, and in view of Remarks 3.5, we have the following.

**4.3 Theorem.** *A commutative GH-algebra is the same thing as a commutative associative algebra over  $\mathbb{R}$  that satisfies SA1–SA7 and is monotone  $\sigma$ -complete. In particular, a commutative GH-algebra is a Banach algebra.*

Let  $X$  be a compact Hausdorff space. As observed in [18],  $C(X, \mathbb{R})$  satisfies all of the synaptic algebra axioms, with the possible exception of the existence of carriers (i.e., SA6). In fact, by [26, Theorem 6.3],  $C(X, \mathbb{R})$  is a synaptic algebra iff  $X$  is basically disconnected and the latter condition is equivalent to the condition that  $C(X, \mathbb{R})$  is monotone  $\sigma$ -complete. As we observed above, if a synaptic algebra is commutative, then it satisfies the CV property (i.e., it is a GH-algebra) iff it is monotone  $\sigma$ -complete. Therefore,  $C(X, \mathbb{R})$  is a synaptic algebra iff it is a (commutative) GH-algebra.

For a commutative GH-algebra we have the following functional representation theorem ([18, Theorem 5.9], [26, Theorem 6.5]). (See [18, Theorem 4.1] for a more general functional representation of a commutative synaptic algebra.)

**4.4 Theorem.** *Suppose that  $A$  is a commutative GH-algebra and let  $X$  be the basically disconnected Stone space of the  $\sigma$ -complete Boolean algebra  $P$ . Then  $C(X, \mathbb{R})$  is a commutative GH-algebra and there exists a GH-isomorphism  $\Psi : A \rightarrow C(X, \mathbb{R})$  such that the restriction  $\psi$  of  $\Psi$  to  $P$  is a Boolean isomorphism of  $P$  onto  $P(X, \mathbb{R})$  as per Stone's theorem.*

**4.5 Theorem.** (i) *Every commutative GH-algebra is a monotone  $\sigma$ -complete order unit normed vector lattice.* (ii) *Conversely, if  $(V, u)$  is a monotone  $\sigma$ -complete order unit normed vector lattice and  $B$  is the Boolean  $\sigma$ -algebra of*

characteristic elements in  $V$ , then a multiplication  $(v, w) \mapsto vw$  can be defined on  $V$  in such a way that  $V$  becomes a commutative Banach GH-algebra with order unit  $u$  and  $B$  is precisely the set  $P$  of projections in  $V$ .

*Proof.* Part (i) follows from Theorems 4.1 and 4.3. To prove (ii), let  $(V, u)$  be a monotone  $\sigma$ -complete order unit normed vector lattice. By Theorem 2.4,  $V$  is a Banach space. By Theorem 2.8, the set  $B$  of characteristic elements of  $V$  forms a Boolean  $\sigma$ -algebra with a basically disconnected Stone space  $X$ ; moreover, there is isomorphism of order unit normed vector lattices,  $\Psi: V \rightarrow C(X, \mathbb{R})$ , of  $V$  onto  $C(X, \mathbb{R})$  such that the restriction  $\psi$  of  $\Psi$  to  $B$  is a Boolean isomorphism of  $B$  onto  $P(X, \mathbb{R})$ . In particular,  $\Psi(u) = \psi(u) = 1$ . By [26, Theorem 6.3],  $C(X, \mathbb{R})$  is a commutative GH-algebra. For  $v, w \in V$ , we define the product  $vw \in V$  by  $vw := \Psi^{-1}(\Psi(v)\Psi(w))$ , whereupon  $V$  is organized into a commutative, associative, partially ordered archimedean real linear algebra with a unity element  $u$  that is also an order unit. Obviously,  $\Psi: V \rightarrow C(X, \mathbb{R})$  is an isomorphism of real linear algebras. Since  $C(X, \mathbb{R})$  is a commutative GH-algebra, so is  $V$ . Also, if  $p \in V$ , then  $p = p^2$  iff  $\Psi(p) = (\Psi(p))^2$  iff  $\Psi(p) \in P(X, \mathbb{R})$  iff  $p \in B$ .  $\square$

## 5 States

Just as is the case for any order unit normed space, a *state* on the synaptic algebra  $A$  is defined to be a linear functional  $\rho: A \rightarrow \mathbb{R}$  that is positive ( $a \in A^+ \Rightarrow \rho(a) \in \mathbb{R}^+$ ) and normalized ( $\rho(1) = 1$ ) [26, Definition 4.5]. The set of states on  $A$  (the *state space* of  $A$ ), which is a convex set, is denoted by  $S(A)$ , and the set of extreme points in  $S(A)$  (*extremal states* on  $A$ ) is denoted by  $Ext(S(A))$ . A state  $\rho \in S(A)$  is said to be  $\sigma$ -additive iff, for every ascending sequence  $(a_n)_{n \in \mathbb{N}}$  in  $A$ ,  $a_n \nearrow a \in A \Rightarrow \rho(a_n) \nearrow \rho(a)$  in  $\mathbb{R}$ . By [26, Theorems 4.6, 4.9]  $S(A)$  determines both the partial order and the order-unit norm on  $A$ .

The next three theorems provide characterizations of extremal states on a commutative synaptic algebra and on a commutative GH-algebra.

**5.1 Theorem.** [26, Theorem 7.1] *Let  $A$  be a commutative synaptic algebra, hence a vector lattice. Then for  $\rho \in S(A)$ , the following conditions are mutually equivalent: (i)  $\rho \in Ext(S(A))$ . (ii)  $\rho: A \rightarrow \mathbb{R}$  is a lattice homomorphism. (iii)  $\rho(a \wedge b) = \min\{\rho(a), \rho(b)\}$  for all  $a, b \in A^+$ .*

**5.2 Theorem.** [26, Theorem 7.2] *Suppose that  $A$  is a commutative GH-algebra,  $X$  is the compact Hausdorff basically disconnected Stone space of the  $\sigma$ -complete Boolean algebra  $P$ ,  $\Psi : A \rightarrow C(X, \mathbb{R})$  is the synaptic isomorphism of Theorem 4.4, and  $\rho \in S(A)$ . Then the following conditions are mutually equivalent:*

- (i)  $\rho \in \text{Ext}(S(A))$ .
- (ii) *There exists  $x \in X$  such that  $\rho(a) = (\Psi(a))(x)$  for all  $a \in A$ .*
- (iii)  $\rho$  is multiplicative, i.e.,  $\rho(ab) = \rho(a)\rho(b)$  for all  $a, b \in A$ .
- (iv)  $\rho(p) \in \{0, 1\}$  for all  $p \in P$ .

**5.3 Theorem.** *Let  $A$  be a commutative GH-algebra. Then, with the notation of Theorem 5.2: (i) There is an affine bijection  $\rho \leftrightarrow \gamma$  between states  $\rho \in S(A)$  and states  $\gamma \in S(C(X, \mathbb{R}))$  such that  $\gamma(f) = \rho(\Psi^{-1}(f))$  for all  $f \in C(X, \mathbb{R})$  and  $\rho(a) = \gamma(\Psi(a))$  for all  $a \in A$ . In particular, if  $\rho \leftrightarrow \gamma$ , then  $\rho$  is extremal iff  $\gamma$  is extremal. (ii) There is a bijective correspondence  $x \leftrightarrow \rho_x$  between points  $x \in X$  and extremal states  $\rho_x$  on  $A$  such that  $\rho_x(a) = (\Psi(a))(x)$  for all  $x \in X$  and all  $a \in A$ .*

*Proof.* Part (i) follows from the fact that  $\Psi : A \rightarrow C(X, \mathbb{R})$  is a synaptic isomorphism.

(ii) According to [26, Theorem 4.10 (iii), (iv)], each  $x \in X$  induces a state  $\gamma_x \in S(C(X, \mathbb{R}))$  such that  $\gamma_x(f) = f(x)$  for all  $f \in C(X, \mathbb{R})$ ; moreover,  $\text{Ext}(S(C(X, \mathbb{R}))) = \{\gamma_x : x \in X\}$ . For each  $x \in X$ , let  $\rho_x$  be the extremal state on  $A$  corresponding to the extremal state  $\gamma_x$  on  $C(X, \mathbb{R})$  according to (i), so that  $\rho_x(a) = \gamma_x(\Psi(a)) = (\Psi(a))(x)$  for all  $a \in A$ . If  $x, y \in X$  and  $\rho_x = \rho_y$ , then  $f(x) = f(y)$  for all  $f \in C(X, \mathbb{R})$ , and since  $C(X, \mathbb{R})$  separates points in  $X$ , it follows that  $x = y$ .  $\square$

**5.4 Definition.** Let  $A$  be a commutative GH-algebra. Then, with the notation of Theorem 5.3, we define  $\hat{a} := \Psi(a) \in C(X, \mathbb{R})$  for all  $a \in A$ . Also, in view of Theorem 5.3 (ii), we can and do identify each point  $x \in X$  with the extremal state  $\rho_x$  on  $A$ , so that  $x(a) = \hat{a}(x)$  for all  $a \in A$  and all  $x \in X$ .

## 6 A Loomis-Sikorski type theorem for GH-algebras

The classical Loomis-Sikorski theorem for Boolean  $\sigma$ -algebras [31, 39] has been extended to  $\sigma$ MV-algebras in [2, 10, 34] (Theorem 6.2 below). In [10], it was also generalized to Dedekind  $\sigma$ -complete abelian  $\ell$ -groups (see also [11, §7.14]). In this section, we shall generalize the Loomis-Sikorski theorem to commutative GH-algebras (Theorem 6.6 below).

We shall be dealing with a nonempty set  $X$  and with the set  $\mathbb{R}^X$  of all functions  $f: X \rightarrow \mathbb{R}$ . The partial order  $\leq$  is defined pointwise on  $\mathbb{R}^X$ , i.e., for  $f, g \in \mathbb{R}^X$ ,  $f \leq g$  iff  $f(x) \leq g(x)$  for all  $x \in X$ . Similarly, the operations  $f \pm g$ ,  $fg$ ,  $\alpha f$  for  $\alpha \in \mathbb{R}$ ,  $\min(f, g)$  and  $\max(f, g)$  are understood to be defined pointwise on  $\mathbb{R}^X$ . The constant functions  $x \mapsto 0$  and  $x \mapsto 1$  will be denoted by  $0, 1 \in \mathbb{R}^X$ . Unless  $X$  is finite, 1 cannot be an order unit in  $\mathbb{R}^X$ ; however,  $\mathbb{R}^X$  is an archimedean vector lattice with  $f \vee g = \max(f, g)$  and  $f \wedge g = \min(f, g)$ , and it is also a commutative linear algebra with unit 1 under the pointwise operations. The set  $X$  may be required to satisfy certain special conditions, e.g., it might be stipulated that  $X$  is a Stone space.

Let  $(f_n)_{n=1}^\infty$  be a sequence in  $\mathbb{R}^X$  that is bounded above in  $\mathbb{R}^X$  (i.e., there exists  $g \in \mathbb{R}^X$  such that  $f_n \leq g$  for all  $n \in \mathbb{N}$ ). Then by Dedekind completeness of  $\mathbb{R}$ , the pointwise supremum of  $(f_n)_{n=1}^\infty$  exists and will be denoted by  $\sup_n f_n \in \mathbb{R}^X$ . Thus,  $(\sup_n f_n)(x) := \sup_n (f_n(x))$  for each  $x \in X$ , and it is clear that  $\sup_n f_n = \bigvee_{n \in \mathbb{N}} f_n$  in the poset  $\mathbb{R}^X$ . Thus,  $\mathbb{R}^X$  is Dedekind  $\sigma$ -complete.

We shall be considering nonempty subsets  $\mathcal{T} \subseteq \mathbb{R}^X$  that satisfy certain prescribed conditions, e.g., regarding their poset structure, being closed under various operations on  $\mathbb{R}^X$ , consisting only of specified kinds of functions, etc.

**6.1 Definition.** A *tribe* on the nonempty set  $X$  is a set  $\mathcal{T} \subseteq \mathbb{R}^X$  that satisfies the following conditions: (1)  $f \in \mathcal{T} \Rightarrow 0 \leq f \leq 1$ . (2)  $0 \in \mathcal{T}$ . (3)  $f \in \mathcal{T} \Rightarrow 1 - f \in \mathcal{T}$ . (4)  $f, g \in \mathcal{T} \Rightarrow \min(f + g, 1) \in \mathcal{T}$ . (5) If  $(f_n)_{n=1}^\infty$  is an ascending sequence of functions in  $\mathcal{T}$  and  $f_n \nearrow f \in \mathbb{R}^X$  pointwise (i.e.,  $f = \sup_n f_n$ ) then  $f \in \mathcal{T}$ .

By [11, Proposition 7.1.6], every tribe is a  $\sigma$ MV-algebra that is closed under pointwise suprema of sequences of its elements.

If  $X$  is any nonempty set, then the set  $\mathcal{T}_e$  of all functions  $f \in \mathbb{R}^X$  such that  $0 \leq f \leq 1$  is a convex tribe and every tribe  $\mathcal{T} \subseteq \mathbb{R}^X$  satisfies  $\mathcal{T} \subseteq \mathcal{T}_e$ .

Moreover, the intersection of any family of tribes (convex tribes)  $\mathcal{T} \subseteq \mathbb{R}^X$  is again a tribe (a convex tribe). Thus, if  $\mathcal{T}_0 \subseteq \mathcal{T}_e$ , then the intersection of all tribes (convex tribes)  $\mathcal{T}$  with  $\mathcal{T}_0 \subseteq \mathcal{T}$  is a tribe (a convex tribe) called the tribe (the convex tribe) *generated by*  $\mathcal{T}_0$ .

For purposes of this paper, by a *functional representation* of a partially ordered algebraic structure  $\mathcal{P}$  we shall mean a subset  $\mathcal{T} \subseteq \mathbb{R}^X$  with partially ordered and algebraic structure corresponding to that of  $\mathcal{P}$  together with a surjective morphism  $h: \mathcal{T} \rightarrow \mathcal{P}$  of  $\mathcal{T}$  onto  $\mathcal{P}$ . For instance, the Loomis-Sikorski functional representation theorem for  $\sigma$ MV-algebras [11, Theorem 1.7.22] is as follows.

**6.2 Theorem.** *For each  $\sigma$ MV-algebra  $M$  there exists a nonempty set  $X$ , a tribe  $\mathcal{T} \subseteq \mathbb{R}^X$ , and a surjective  $\sigma$ MV-homomorphism  $h: \mathcal{T} \rightarrow M$  from  $\mathcal{T}$  onto  $M$ .*

We note that Theorem 4.4 yields a functional representation for a commutative GH-algebra  $A$  in terms of a basically disconnected Stone space  $X$  and the commutative GH-algebra  $C(X, \mathbb{R}) \subseteq \mathbb{R}^X$ ; indeed it provides a GH-isomorphism  $\Psi^{-1}: C(X, \mathbb{R}) \rightarrow A$ . However, this representation may suffer from the defect that there may be bounded sequences  $(f_n)_{n \in \mathbb{N}}$  in  $C(X, \mathbb{R})$  such that the supremum  $\bigvee_{n \in \mathbb{N}} f_n$  in  $C(X, \mathbb{R})$  exists but is not the pointwise supremum  $\sup_n f_n$ . In the Loomis-Sikorski functional representation of  $A$  (Theorem 6.6 below), this defect is corrected by employing the following definition (which is an extension of the notion of a g-tribe [11, p. 465]).

**6.3 Definition.** A *gh-tribe* on the nonempty set  $X$  is a set  $\mathcal{T} \subseteq \mathbb{R}^X$  such that:

- (1) If  $f \in \mathcal{T}$ , then  $f$  is bounded, i.e., there exist  $\alpha, \beta \in \mathbb{R}$  with  $\alpha \leq f(x) \leq \beta$  for all  $x \in X$ .
- (2)  $0, 1 \in \mathcal{T}$ .
- (3)  $f + g \in \mathcal{T}$  whenever  $f, g \in \mathcal{T}$ .
- (4) For every  $\alpha \in \mathbb{R}$  and every  $f \in \mathcal{T}$ ,  $\alpha f \in \mathcal{T}$ .
- (5) If  $(f_n)_{n=1}^\infty$  is a sequence of functions in  $\mathcal{T}$  for which there exists  $f \in \mathcal{T}$  with  $f_n \leq f$  for all  $n \in \mathbb{N}$ , then the pointwise supremum  $\sup_n f_n$  belongs to  $\mathcal{T}$ .



Let  $\mathcal{T}$  be a gh-tribe on  $X$ . In (5), it is clear that the pointwise supremum  $\sup_n f_n$  is also the supremum  $\bigvee_{n \in \mathbb{N}} f_n$  in  $\mathcal{T}$ , so  $\mathcal{T}$  is Dedekind  $\sigma$ -complete. Suppose  $f, g \in \mathcal{T}$ . By (1), there exists  $\beta \in \mathbb{R}$  with  $f(x), g(x) \leq \beta$  for all  $x \in X$ , so  $f, g \leq \beta 1$ . Thus the pointwise supremum  $\max(f, g)$  of  $f$  and  $g$  belongs to  $\mathcal{T}$  and it is the supremum  $f \vee g$  of  $f$  and  $g$  in  $\mathcal{T}$ . As  $f \mapsto -f$  is an involution on  $\mathcal{T}$ , it follows that the infimum  $f \wedge g$  exists in  $\mathcal{T}$ , and  $f \wedge g = \min(f, g)$ , whence  $\mathcal{T}$  is a vector lattice. By (1), 1 is an order unit in  $\mathcal{T}$ , and since  $\mathbb{R}^X$  is archimedean, so is  $\mathcal{T}$ . Therefore, in view of Theorem 2.6, every gh-tribe is a Dedekind  $\sigma$ -complete Banach order unit normed vector lattice with order unit 1.

Let  $X$  be any nonempty set. Then the set  $\mathcal{T}_b$  of all bounded functions in  $\mathbb{R}^X$  is a gh-tribe on  $X$  and every gh-tribe  $\mathcal{T}$  on  $X$  satisfies  $\mathcal{T} \subseteq \mathcal{T}_b$ . Moreover, the intersection of any family of gh-tribes on  $X$  is again a gh-tribe on  $X$ . Thus, if  $\mathcal{T}_0 \subseteq \mathcal{T}_b$ , then the intersection of all gh-tribes  $\mathcal{T}$  with  $\mathcal{T}_0 \subseteq \mathcal{T}$  is a gh-tribe called the gh-tribe *generated by*  $\mathcal{T}_0$ .

Clearly, every order unit normed vector lattice  $\mathcal{T} \subseteq \mathbb{R}^X$  with order unit 1 satisfies  $\mathcal{T} \subseteq \mathcal{T}_b$ . Moreover, the intersection of any family of such order unit normed vector lattices  $\mathcal{T} \subseteq \mathbb{R}^X$  is again an order unit normed vector lattice with order unit 1. If  $\mathcal{T}_0 \subseteq \mathcal{T}_b$ , then the intersection of all order unit normed vector lattices  $\mathcal{T}$  with order unit 1 such that  $\mathcal{T}_0 \subseteq \mathcal{T}$  is called the order unit normed vector lattice *generated by*  $\mathcal{T}_0$ .

Let  $\mathcal{T}$  be a gh-tribe on  $X$ . Since  $\mathcal{T}$  is a Dedekind  $\sigma$ -complete Banach order unit normed vector lattice, we infer from Theorem 4.5 that there exists a multiplication operation  $(f, g) \mapsto fg$  on  $\mathcal{T}$  such that  $\mathcal{T}$  forms a commutative Banach GH-algebra and the set of characteristic elements in  $\mathcal{T}$  coincides with the Boolean  $\sigma$ -algebra  $\mathcal{P}$  of projections (idempotents) in  $\mathcal{T}$ .

**6.4 Lemma.** *Let  $\mathcal{T}$  be a gh-tribe on  $X$  and let  $f, g, p \in \mathcal{T}$ . Then: (i)  $|f| \in \mathcal{T}$  is the pointwise absolute value. (ii) Norm limits in  $\mathcal{T}$  are pointwise limits. (iii)  $p \in \mathcal{P} \Leftrightarrow p(x) \in \{0, 1\}$  for all  $x \in X$ . (iv) Multiplication on  $\mathcal{T}$  is pointwise.*

*Proof.* (i)  $|f|(x) = (f \vee (-f))(x) = \max(f(x), -f(x)) = |f(x)|$  for all  $x \in X$ .

(ii) Suppose that  $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}$ . By Lemma 2.7 (ii) and (i),  $\lim_{n \rightarrow \infty} f_n = f$  iff  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in X$ .

(iii) We have  $p \in \mathcal{P}$  iff  $p \wedge (1 - p) = 0$  iff  $\min(p(x), 1 - p(x)) = 0$  for all  $x \in X$  iff (iii) holds.

(iv) Let  $p, q \in \mathcal{P}$ . Then by Lemma 3.3 and (iii),

$$(pq)(x) = (p \wedge q)(x) = \min(p(x), q(x)) = p(x)q(x) \text{ for all } x \in X.$$

Therefore,  $pq$  is the pointwise product of  $p$  and  $q$ , and it follows that the product of simple elements in  $\mathcal{T}$  is the pointwise product. There are ascending sequences  $(s_n)_{n \in \mathbb{N}}$  and  $(t_n)_{n \in \mathbb{N}}$  of simple elements in  $\mathcal{T}$  such that  $\lim_{n \rightarrow \infty} s_n = f$  and  $\lim_{n \rightarrow \infty} t_n = g$ , whence  $fg = \lim_{n \rightarrow \infty} (s_n t_n)$ , and by (ii),

$$\begin{aligned} (fg)(x) &= \lim_{n \rightarrow \infty} (s_n t_n)(x) = \lim_{n \rightarrow \infty} (s_n(x) t_n(x)) = \left( \lim_{n \rightarrow \infty} s_n(x) \right) \left( \lim_{n \rightarrow \infty} t_n(x) \right) \\ &= f(x)g(x) \text{ for all } x \in X. \end{aligned} \quad \square$$

For a gh-tribe  $\mathcal{T}$  on  $X$  we define

$$\mathcal{B}(\mathcal{T}) := \{D \subseteq X : \chi_D \in \mathcal{T}\}$$

and note that  $\mathcal{B}(\mathcal{T})$  is a  $\sigma$ -field of subsets of  $X$ . Also, the unit interval  $\mathcal{T}[0, 1] := \{f \in \mathcal{T} : 0 \leq f \leq 1\} \subseteq \mathbb{R}^X$  is a convex  $\sigma$ MV-algebra, which is also a convex tribe. According to [12, Theorem 7.4], the following statement, which is an extension of [7] for tribes, holds (see also [36]).

**6.5 Theorem.** *Let  $\mathcal{T}$  be a gh-tribe on  $X$ . Then every  $f \in \mathcal{T}$  is  $\mathcal{B}(\mathcal{T})$ -measurable, and for every  $\sigma$ -additive state  $\rho$  on  $\mathcal{T}$ ,  $\rho(f) = \int_X f(x) \mu(dx)$ , where  $\mu$  is the probability measure on  $\mathcal{B}(\mathcal{T})$  defined by  $\mu(D) := \rho(\chi_D)$ .*

We can now state and prove the Loomis-Sikorski theorem for commutative GH-algebras.

**6.6 Theorem. (The Loomis-Sikorski theorem.)** *For every commutative GH-algebra  $A$  there exists a gh-tribe  $\mathcal{T}$  of bounded functions on a compact Hausdorff space  $X$  and a surjective morphism of GH-algebras  $\hat{h}$  from  $\mathcal{T}$  onto  $A$ .*

*Proof.* We begin by proving a version of Theorem 6.2 involving convexity. The effect algebra  $E \subseteq A$  is a convex  $\sigma$ MV-algebra, and we let  $X$  be the basically disconnected Stone space of the  $\sigma$ -complete Boolean algebra  $P$  of projections in  $A$ . As per Definition 5.4, we shall identify  $X$  with the set  $\text{Ext}(S(A))$  of all extremal states on  $A$ . By Theorem 4.4, there exists a GH-isomorphism  $\Psi : A \rightarrow C(X, \mathbb{R})$  that restricts to a Boolean isomorphism  $\psi$  of  $P$  onto the Boolean algebra  $P(X, \mathbb{R}) \subseteq C(X, \mathbb{R})$ . As in Definition 5.4, for  $a \in A$ , we write  $\hat{a} := \Psi(a)$ , so that  $\hat{a}(x) = x(a)$  for all  $x \in X$ .

Let  $\mathcal{T}_1$  be the convex tribe generated by  $C(X, \mathbb{R})[0, 1] = \{\hat{a} \in C(X, \mathbb{R}) : a \in E\}$ . If  $f \in \mathcal{T}_1$ , define  $N(f) := \{x \in X : f(x) \neq 0\}$ .

Let  $\mathcal{T}'_1$  be the set of functions  $f \in \mathcal{T}_1$  with the property that, for some  $a \in E$ ,  $N(f - \hat{a})$  is a meager set (a countable union of nowhere dense sets). If  $a_1$  and  $a_2$  are two elements of  $E$  such that  $N(f - \hat{a}_1)$  and  $N(f - \hat{a}_2)$  are meager sets, then  $N(\hat{a}_1 - \hat{a}_2) \subseteq N(f - \hat{a}_1) \cup N(f - \hat{a}_2)$  is a meager set. By the Baire theorem, a non-empty open subset of a compact Hausdorff space cannot be a meager set, whence  $\hat{a}_1 = \hat{a}_2$ , and it follows that such an  $a \in E$  is unique. Therefore the mapping  $h : \mathcal{T}'_1 \rightarrow E$  defined by  $h(f) = a$  iff  $a \in E$  and  $N(f - \hat{a})$  is meager is well defined and maps  $\mathcal{T}'_1$  onto  $E$ .

Proceeding as in the proof of [11, Theorem 7.1.22], we deduce that  $\mathcal{T}'_1$  is a tribe. Clearly, if  $0 \leq \alpha \leq 1$  and  $N(f - \hat{a})$  is meager, then  $N(\alpha f - \alpha \hat{a})$  is meager, so we may infer that  $\mathcal{T}'_1$  is a convex tribe that contains  $C(X, \mathbb{R})[0, 1]$  and that the mapping  $h$  preserves the convex structure. From this it follows that  $\mathcal{T}_1 \subseteq \mathcal{T}'_1$ , and hence  $\mathcal{T}_1 = \mathcal{T}'_1$ . Moreover, the mapping  $h : \mathcal{T}_1 \rightarrow E$  corresponds to the surjective  $\sigma$ MV-algebra homomorphism of Theorem 6.2.

Now let  $\mathcal{T}$  be the gh-tribe of functions  $f : X \rightarrow \mathbb{R}$  generated by  $C(X, \mathbb{R})$ . We note that, by [11, Proposition 7.1.25],  $\mathcal{T}$  is the set of all bounded Baire measurable functions on  $X$ . We claim that  $\mathcal{T}_1 = \mathcal{T}[0, 1] = \{f \in \mathcal{T} : 0 \leq f \leq 1\}$ . From  $C(X, \mathbb{R})[0, 1] \subseteq C(X, \mathbb{R})$  we deduce that

$$\mathcal{T}_1 \subseteq \mathcal{T}[0, 1]. \quad (1)$$

Let  $\mathcal{V}$  be the order unit normed vector lattice generated by  $\mathcal{T}_1$ . By [28, Theorem 16.9],  $\mathcal{V}$  is monotone  $\sigma$ -complete since  $\mathcal{T}_1$  is. It follows that  $\mathcal{V}$  is a gh-tribe, and from (1) we conclude that  $\mathcal{V} \subseteq \mathcal{T}$ . Since  $C(X, \mathbb{R}) \subseteq \mathcal{V}$ , we conclude that  $\mathcal{V} = \mathcal{T}$ , and hence  $\mathcal{T}_1 = \mathcal{T}[0, 1]$ .

Using arguments extracted from Mundici's proof [35] [11, Corollary 5.3.8] of the categorical equivalence of unital abelian  $\ell$ -groups and MV-algebras, we conclude that there exists an extension  $\hat{h} : \mathcal{T} \rightarrow A$  of the mapping  $h$  such that  $\hat{h}$  is an  $\ell$ -group homomorphism, i.e., (i)  $\hat{h}(f) \pm \hat{h}(g) = \hat{h}(f \pm g)$ , (ii)  $\hat{h}(\max(f, g)) = \hat{h}(f) \vee \hat{h}(g)$ , and (iii)  $\hat{h}(1) = 1$ . To prove that  $\hat{h}(\sup_n f_n) = \sup_n \hat{h}(f_n)$  whenever  $\sup_n f_n \in \mathcal{T}$  we use the same setup as in the proof of [11, Theorem 7.1.24]. By [26, Lemma 4.2],  $\hat{h}$  is a positive linear mapping from  $\mathcal{T}$  onto  $A$ . Since every extremal state is multiplicative, we have for all  $a, b \in E$ ,  $\hat{a}\hat{b} = \hat{a}\hat{b}$ , and owing to the inequalities

$$|fg - \hat{a}\hat{b}| = |fg - \hat{a}\hat{b}| \leq |fg - f\hat{b}| + |f\hat{b} - \hat{a}\hat{b}| = |f| |g - \hat{b}| + |f - \hat{a}| |\hat{b}|,$$

$N(fg - \hat{a}\hat{b})$  is a meager set provided that  $N(f - \hat{a})$  and  $N(g - \hat{b})$  are meager. From this it follows that  $h(fg) = h(f)h(g)$ , so that  $h : \mathcal{T}_1 \rightarrow E$  preserves

multiplication. Considering first  $f, g \in \mathcal{T}^+$ , and then taking into account the decomposition  $f = f^+ - f^-$  for all  $f \in \mathcal{T}$ , it is easy to see that  $\hat{h}$  preserves multiplication. We conclude that  $\hat{h}$  is a GH-algebra morphism from the gh-tribe  $\mathcal{T}$  onto  $A$ .  $\square$

We define the *canonical representation* of the commutative GH-algebra  $A$  to be the triple  $(X, \mathcal{T}, \hat{h})$  in Theorem 6.6. Notice that the canonical representation is *regular* in the sense that  $h(f) = 0$  iff  $h(\chi_{N(f)}) = 0$ .

## References

- [1] E.M. Alfsen: Compact Convex Sets and Boundary Integrals, Springer-Verlag, Heidelberg, New York, 1971.
- [2] G. Barbieri, H. Weber: Measures on clans and on MV-algebras, in: E. Pap (Ed.), Handbook of Measure Theory, vol. II, Elsevier, Amsterdam, 2002, Chapter 22.
- [3] Garrett Birkhoff, Lattice Theory, A.M.S. Colloquium Publications, XXV, Providence, R.I., 1967.
- [4] P. Busch, P. Lahti, J. J. Pekka, P. Mittelstaedt: The quantum theory of measurement, Second edition. Lecture Notes in Physics. New Series m: Monographs, 2. Springer-Verlag, Berlin, 1996.
- [5] S. Gudder, E. Beltrametti, S. Bugajski, S. Pulmannová: Convex and linear effect algebras, Rep. Math. Phys. **44** (1999) 359–379.
- [6] S. Bugajski, S. Gudder, S. Pulmannová: Convex effect algebras, state ordered effect algebras and ordered linear spaces, Rep. Math. Phys. **45** (2005), 371–388.
- [7] D. Butnariu, E.P. Klement: Triangular norm-based measures and their Markov kernel representation, J. Math. Anal. Appl. **162** (1991) 111–143.
- [8] C.C. Chang: Algebraic analysis of many-valued logic, Trans. Amer. Math. Soc. **88** (1958) 467–490.

- [9] R. Cristescu: Ordered Vector Spaces and Linear Operators, Editura Academii Bucuresti and ABACUS Press, Tunbridge Wells, Kent, 1976.
- [10] A. Dvurečenskij: Loomis-Sikorski theorem for  $\sigma$ -complete MV-algebras and  $\ell$ -groups, J. Austral. Math. Soc. **Ser. A 68** (2000) 261–277.
- [11] A. Dvurečenskij, S. Pulmannová: New Trends in Quantum Structures, Inter Science, Bratislava and Kluwer, Dordrecht, 2000.
- [12] A. Dvurečenskij, S. Pulmannová: Conditional probability on  $\sigma$ -MV-algebras, Fuzzy Sets and Syst. **155** (2005), 102–118.
- [13] D.J. Foulis, M.K. Bennett: Effect algebras and unsharp quantum logics, Found. Phys. **24** (1994) 1331–1352.
- [14] D.J. Foulis: Synaptic algebras, Math. Slovaca **60** (2010) 631–654.
- [15] D.J. Foulis, S. Pulmannová: Spectral resolution in an order-unit space, Rep. Math. Phys. **62**, no. 3 (2008) 323–344.
- [16] D.J. Foulis, S. Pulmannová: Generalized Hermitian algebras, Internat. J. Theoret. Phys. **48** (2009) 1320–1333.
- [17] D. Foulis, S. Pulmannová: Spin factors as generalized Hermitian algebras, Found. Phys. **39** (2009) 237–255.
- [18] D.J. Foulis, S. Pulmannová: Projections in a synaptic algebra, Order **27** (2010) 235–257.
- [19] D.J. Foulis, S. Pulmannová: Regular elements in generalized Hermitian algebra, Math. Slovaca **61** (2011) 155–172.
- [20] D.J. Foulis, S. Pulmannová: Type-decomposition of a synaptic algebra, Found. Phys. **43**, no. 8 (2013) 948–968.
- [21] D.J. Foulis, S. Pulmannová: Symmetries in a synaptic algebra, Math. Slovaca **64**, no. 3 (2014) 751–776.
- [22] D.J. Foulis, S. Pulmannová: Commutativity in synaptic algebras, Math. Slovaca **66** (2016) 469–482.

- [23] D.J. Foulis, S. Pulmannová: Handelman's theorem for an order unit normed space, arXiv:1609.08014v1[math.FA] 23 Sep 2016.
- [24] D.J. Foulis, A. Jenčová, S. Pulmannová: Two projections in a synaptic algebra, *Linear Algebra Appl.* **478** (2015) 163–287.
- [25] D.J. Foulis, A. Jenčová, S. Pulmannová: A projection and an effect in a synaptic algebra, *Linear Algebra Appl.* **485** (2015) 417–441.
- [26] D.J. Foulis, A. Jenčová, S. Pulmannová: States and synaptic algebras, *Rep. Math. Phys.*, to appear, arXiv:1606.08229[math-ph].
- [27] D.J. Foulis, A. Jenčová, S. Pulmannová: Vector lattices in synaptic algebras, to appear, arXiv:1605.06987[math.RA].
- [28] K.R. Goodearl: Partially Ordered Abelian Groups with Interpolation, *Math. Surveys and Monographs* No. 20, AMs, Providence, 1986.
- [29] D. Handelman: Rings with involution as partially ordered abelian groups, *Rocky Mt. J. Math.* **11** (1981) 337–381.
- [30] G. Kalmbach: Orthomodular Lattices, Academic Press, London, New York, 1983.
- [31] L.H. Loomis: On the representation of  $\sigma$ -complete Boolean algebras, *Bull. Amer. Math. Soc.* **53** (1947) 757–760.
- [32] G.W. Mackey: Mathematical Foundations of Quantum Mechanics, Benjamin, New York, 1963.
- [33] K. McCrimmon: A taste of Jordan algebras, Universitext, Springer-Verlag, New York, 2004.
- [34] D. Mundici: Tensor product and the Loomis-Sikorski theorem for MV-algebras, *Adv. Appl. Math.* **22** (1999) 227–248.
- [35] D. Mundici: Interpretation of AF-C\*-algebras in Łukasiewicz sentential calculus, *J. Funct. Anal.* **65** (1986) 15–63.
- [36] S. Pulmannová: Spectral resolutions in Dedekind  $\sigma$ -complete  $\ell$ -groups, *J. Math. Anal. Appl.* **309** (2005) 322–335.

- [37] S. Pulmannová: A note on ideals in synaptic algebras, Math. Slovaca **62**, no. 6 (2012), 1091–1104.
- [38] H.H. Schaefer: Banach Lattices and Positive operators, Springer, Berlin Heidelberg New York, 1974.
- [39] R. Sikorski: Boolean Algebras, Springer, Berlin-Heidelberg-New York, 1964.
- [40] V.S. Varadarajan: Geometry of Quantum Theory, Springer-Verlag, New York-Berlin, 1985.